

Weighted Endpoint Estimates for Commutators of Calderón-Zygmund Operators

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Abstract Let $\delta \in (0, 1]$ and T be a δ -Calderón-Zygmund operator. Let w be in the Muckenhoupt class $A_{1+\delta/n}(\mathbb{R}^n)$ satisfying $\int_{\mathbb{R}^n} \frac{w(x)}{1+|x|^n} dx < \infty$. When $b \in \text{BMO}(\mathbb{R}^n)$, it is well known that the commutator $[b, T]$ is not bounded from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ if b is not a constant function. In this article, the authors find out a proper subspace $\mathcal{BMO}_w(\mathbb{R}^n)$ of $\text{BMO}(\mathbb{R}^n)$ such that, if $b \in \mathcal{BMO}_w(\mathbb{R}^n)$, then $[b, T]$ is bounded from the weighted Hardy space $H_w^1(\mathbb{R}^n)$ to the weighted Lebesgue space $L_w^1(\mathbb{R}^n)$. Conversely, if $b \in \text{BMO}(\mathbb{R}^n)$ and the commutators of the classical Riesz transforms $\{[b, R_j]\}_{j=1}^n$ are bounded from $H_w^1(\mathbb{R}^n)$ into $L_w^1(\mathbb{R}^n)$, then $b \in \mathcal{BMO}_w(\mathbb{R}^n)$.

1 Introduction

Given a function b locally integrable on \mathbb{R}^n and a classical Calderón-Zygmund operator T , we consider the linear commutator $[b, T]$ defined by setting, for smooth, compactly supported functions f ,

$$[b, T](f) = bT(f) - T(bf).$$

A classical result of Coifman et al. [4] states that the commutator $[b, T]$ is bounded on $L^p(\mathbb{R}^n)$ for $p \in (1, \infty)$, when $b \in \text{BMO}(\mathbb{R}^n)$. Moreover, their proof does not rely on a weak type $(1, 1)$ estimate for $[b, T]$. Indeed, this operator is more singular than the associated Calderón-Zygmund operator since it fails, in general, to be of weak type $(1, 1)$, when b is in $\text{BMO}(\mathbb{R}^n)$. Moreover, Harboure et al. [7, Theorem (3.1)] showed that $[b, T]$ is bounded from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ if and only if b equals to a constant almost everywhere. Although the commutator $[b, T]$ does not map continuously, in general, $H^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$, following Pérez [11], one can find a subspace $\mathcal{H}_b^1(\mathbb{R}^n)$ of $H^1(\mathbb{R}^n)$ such that $[b, T]$ maps continuously $\mathcal{H}_b^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$. Very recently, Ky [10] found the *largest subspace* of $H^1(\mathbb{R}^n)$ such that all commutators $[b, T]$ of Calderón-Zygmund operators are bounded from this subspace into $L^1(\mathbb{R}^n)$. More precisely, it was showed in [10] that there exists a

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bilinear operators $\mathfrak{R} := \mathfrak{R}_T$ mapping continuously $H^1(\mathbb{R}^n) \times \text{BMO}(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ such that, for all $(f, b) \in H^1(\mathbb{R}^n) \times \text{BMO}(\mathbb{R}^n)$, we have

$$(1.1) \quad [b, T](f) = \mathfrak{R}(f, b) + T(\mathfrak{S}(f, b)),$$

where \mathfrak{S} is a bounded bilinear operator from $H^1(\mathbb{R}^n) \times \text{BMO}(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ which is independent of T . The bilinear decomposition (1.1) allows ones to give a general overview of all known endpoint estimates; see [10] for the details.

For the weighted case, when $b \in \text{BMO}(\mathbb{R}^n)$, Álvarez et al. [1] proved that the commutator $[b, T]$ is bounded on the weighted Lebesgue space $L_w^p(\mathbb{R}^n)$ with $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}^n)$, where $A_p(\mathbb{R}^n)$ denotes the class of Muckenhoupt weights. Similar to the unweighted case, $[b, T]$ may not be bounded from the weighted Hardy space $H_w^1(\mathbb{R}^n)$ into the weighted Lebesgue space $L_w^1(\mathbb{R}^n)$ if b is not a constant function. Thus, a natural question is whether there exists a non-trivial subspace of $\text{BMO}(\mathbb{R}^n)$ such that, when b belongs to this subspace, the commutator $[b, T]$ is bounded from $H_w^1(\mathbb{R}^n)$ to $L_w^1(\mathbb{R}^n)$.

The purpose of the present paper is to give an answer for the above question. To this end, we first recall the definition of the Muckenhoupt weights. A non-negative measurable function w is said to belong to the *class of Muckenhoupt weight* $A_q(\mathbb{R}^n)$ for $q \in [1, \infty)$, denoted by $w \in A_q(\mathbb{R}^n)$ if, when $q \in (1, \infty)$,

$$(1.2) \quad [w]_{A_q(\mathbb{R}^n)} := \sup_{B \subset \mathbb{R}^n} \frac{1}{|B|} \int_B w(x) dx \left\{ \frac{1}{|B|} \int_B [w(y)]^{-q'/q} dy \right\}^{q/q'} < \infty,$$

where $1/q + 1/q' = 1$, or, when $q = 1$,

$$(1.3) \quad [w]_{A_1(\mathbb{R}^n)} := \sup_{B \subset \mathbb{R}^n} \frac{1}{|B|} \int_B w(x) dx \left(\text{ess sup}_{y \in B} [w(y)]^{-1} \right) < \infty.$$

Here the suprema are taken over all balls $B \subset \mathbb{R}^n$. Let

$$A_\infty(\mathbb{R}^n) := \bigcup_{q \in [1, \infty)} A_q(\mathbb{R}^n).$$

Let $w \in A_\infty(\mathbb{R}^n)$ and $q \in (0, \infty]$. If $q \in (0, \infty)$, then we let $L_w^q(\mathbb{R}^n)$ be the space of all measurable functions f such that

$$(1.4) \quad \|f\|_{L_w^q(\mathbb{R}^n)} := \left\{ \int_{\mathbb{R}^n} |f(x)|^q w(x) dx \right\}^{1/q} < \infty.$$

When $q = \infty$, $L_w^\infty(\mathbb{R}^n)$ is defined to be the same as $L^\infty(\mathbb{R}^n)$ and, for any $f \in L_w^\infty(\mathbb{R}^n)$, let

$$\|f\|_{L_w^\infty(\mathbb{R}^n)} := \|f\|_{L^\infty(\mathbb{R}^n)}.$$

Let ϕ be a function in the Schwartz class, $\mathcal{S}(\mathbb{R}^n)$, satisfying $\phi(x) = 1$ for all $x \in B(0, 1)$. The *maximal function* of a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ is defined by

$$(1.5) \quad \mathcal{M}_\phi f := \sup_{t \in (0, \infty)} |f * \phi_t|,$$

where $\phi_t(\cdot) := \frac{1}{t^n} \phi(t^{-1}\cdot)$ for all $t \in (0, \infty)$. Then the *weighted Hardy space* $H_w^1(\mathbb{R}^n)$ is defined as the space of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{H_w^1(\mathbb{R}^n)} := \|\mathcal{M}_\phi f\|_{L_w^1(\mathbb{R}^n)} < \infty;$$

see [5].

Notice that $\|\cdot\|_{H_w^1(\mathbb{R}^n)}$ defines a norm on $H_w^1(\mathbb{R}^n)$, whose size depends on the choice of ϕ , but the space $H_w^1(\mathbb{R}^n)$ is independent of this choice.

Definition 1.1. Let $w \in A_\infty(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \frac{w(x)}{1+|x|^n} dx < \infty$. A locally integrable function b is said to be in $\mathcal{BMO}_w(\mathbb{R}^n)$ if

$$(1.6) \quad \|b\|_{\mathcal{BMO}_w(\mathbb{R}^n)} := \sup_B \left\{ \int_{B^c} \frac{w(x)}{|x - x_B|^n} dx \frac{1}{w(B)} \int_B |b(x) - b_B| dx \right\} < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$ and $B^c := \mathbb{R}^n \setminus B$. Here and hereafter, x_B denotes the center of ball B ,

$$w(B) := \int_B w(x) dx \quad \text{and} \quad b_B := \frac{1}{|B|} \int_B b(x) dx.$$

It should be pointed out that the space $\mathcal{BMO}_w(\mathbb{R}^n)$ has been considered first by Bloom [2] when studying the pointwise multipliers of weighted BMO spaces (see also [14]).

Recall that a locally integrable function b is said to be in $\text{BMO}(\mathbb{R}^n)$ if

$$\|b\|_{\text{BMO}(\mathbb{R}^n)} := \sup_B \frac{1}{|B|} \int_B |b(x) - b_B| dx < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$.

Remark 1.2. (i) $\mathcal{BMO}_w(\mathbb{R}^n) \subset \text{BMO}(\mathbb{R}^n)$ and the inclusion is continuous (see Proposition 2.1 of Section 2).

(ii) It is easy to show that, when $n = 1$, $w(x) := |x|^{-1/2} \in A_1(\mathbb{R})$ and $\int_{\mathbb{R}} \frac{w(x)}{1+|x|} dx < \infty$. Let

$$f(x) := \begin{cases} |1 - x|, & |x| \leq 1, \\ 0, & |x| > 1. \end{cases}$$

Then $f \in \mathcal{BMO}_w(\mathbb{R}^n)$, which implies that $\mathcal{BMO}_w(\mathbb{R}^n)$ is not a trivial function space.

To state our main results, we first recall the definition of Calderón-Zygmund operators. For $\delta \in (0, 1]$, a linear operator T is called a δ -Calderón-Zygmund operator if T is a linear bounded operator on $L^2(\mathbb{R}^n)$ and there exist a kernel K on $(\mathbb{R}^n \times \mathbb{R}^n) \setminus \{(x, x) : x \in \mathbb{R}^n\}$ and a positive constant C such that, for all $x, y, z \in \mathbb{R}^n$,

$$|K(x, y)| \leq \frac{C}{|x - y|^n} \quad \text{if} \quad x \neq y,$$

$$|K(x, y) - K(x, z)| + |K(y, x) - K(z, x)| \leq C \frac{|y - z|^\delta}{|x - y|^{n+\delta}} \quad \text{if } |x - y| > 2|y - z|$$

and, for all $f \in L^2(\mathbb{R}^n)$ with compact support and $x \notin \text{supp}(f)$,

$$Tf(x) = \int_{\text{supp}(f)} K(x, y)f(y) dy.$$

The main result of this paper is the following theorem.

Theorem 1.3. *Let $\delta \in (0, 1]$, $w \in A_{1+\delta/n}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \frac{w(x)}{1+|x|^n} dx < \infty$ and $b \in \text{BMO}(\mathbb{R}^n)$. Then the following two statements are equivalent:*

- (i) *for every δ -Calderón-Zygmund operator T , the commutator $[b, T]$ is bounded from $H_w^1(\mathbb{R}^n)$ into $L_w^1(\mathbb{R}^n)$;*
- (ii) *$b \in \mathcal{BMO}_w(\mathbb{R}^n)$.*

Remark 1.4. When $w(x) \equiv 1$ for all $x \in \mathbb{R}^n$, we see that $\int_{\mathbb{R}^n} \frac{1}{1+|x|^n} dx = \infty$ and hence, in this case, $\mathcal{BMO}_w(\mathbb{R}^n)$ can be seen as a zero space in $\text{BMO}(\mathbb{R}^n)$. In this case, Theorem 1.3 coincides with the result in [7].

The next theorem gives a sufficient condition of the boundedness of $[b, T]$ on $H_w^1(\mathbb{R}^n)$. Recall that, for $w \in A_p(\mathbb{R}^n)$ with $p \in (1, \infty)$ and $q \in [p, \infty]$, a measurable function a is called an $(H_w^1(\mathbb{R}^n), q)$ -atom related to a ball $B \subset \mathbb{R}^n$ if

- (i) $\text{supp } a \subset B$,
- (ii) $\int_{\mathbb{R}^n} a(x) dx = 0$,
- (iii) $\|a\|_{L_w^q(\mathbb{R}^n)} \leq [w(B)]^{1/q-1}$

and also that $T^*1 = 0$ means $\int_{\mathbb{R}^n} Ta(x) dx = 0$ holds true for all $(H_w^1(\mathbb{R}^n), q)$ -atoms a .

Theorem 1.5. *Let $\delta \in (0, 1]$, T be a δ -Calderón-Zygmund operator, $w \in A_{1+\delta/n}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \frac{w(x)}{1+|x|^n} dx < \infty$ and $b \in \mathcal{BMO}_w(\mathbb{R}^n)$. If $T^*1 = 0$, then the commutator $[b, T]$ is bounded on $H_w^1(\mathbb{R}^n)$, namely, there exists a positive constant C such that, for all $f \in H_w^1(\mathbb{R}^n)$,*

$$\|[b, T](f)\|_{H_w^1(\mathbb{R}^n)} \leq C \|f\|_{H_w^1(\mathbb{R}^n)}.$$

Finally we make some conventions on notation. Throughout the whole article, we denote by C a *positive constant* which is independent of the main parameters, but it may vary from line to line. The *symbol* $A \lesssim B$ means that $A \leq CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \sim B$. For any measurable subset E of \mathbb{R}^n , we denote by E^c the set $\mathbb{R}^n \setminus E$ and its *characteristic function* by χ_E . We also let $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$.

2 Proofs of Theorems 1.3 and 1.5

We begin with pointing out that, if $w \in A_\infty(\mathbb{R}^n)$, then there exist $p, r \in (1, \infty)$ such that $w \in A_p(\mathbb{R}^n) \cap RH_r(\mathbb{R}^n)$, where $RH_r(\mathbb{R}^n)$ denotes the *reverse Hölder class* of weights w satisfying that there exists a positive constant C such that

$$\left(\frac{1}{|B|} \int_B [w(x)]^r dx \right)^{1/r} \leq C \frac{1}{|B|} \int_B w(x) dx$$

for every ball $B \subset \mathbb{R}^n$. Moreover, there exist positive constants $C_1 \leq C_2$, depending on $[w]_{A_\infty(\mathbb{R}^n)}$, such that, for any measurable sets $E \subset B$,

$$(2.1) \quad C_1 \left(\frac{|E|}{|B|} \right)^p \leq \frac{w(E)}{w(B)} \leq C_2 \left(\frac{|E|}{|B|} \right)^{(r-1)/r}.$$

In order to prove Theorems 1.3 and 1.5, we need the following proposition and several technical lemmas.

Proposition 2.1. *Let $w \in A_\infty(\mathbb{R}^n)$. Then there exists a positive constant C such that, for any $f \in \mathcal{BMO}_w(\mathbb{R}^n)$,*

$$\|f\|_{\mathcal{BMO}(\mathbb{R}^n)} \leq C \|f\|_{\mathcal{BMO}_w(\mathbb{R}^n)}.$$

Proof. By (2.1), for any ball $B \subset \mathbb{R}^n$, we have

$$\begin{aligned} \int_{B^c} \frac{w(x)}{|x - x_B|^n} dx \frac{1}{w(B)} &\geq \int_{2B \setminus B} \frac{w(x)}{|x - x_B|^n} dx \frac{1}{w(B)} \\ &\geq \frac{w(2B \setminus B)}{|2B|} \frac{1}{w(B)} \\ &\gtrsim \frac{1}{|B|}. \end{aligned}$$

This proves that $\|f\|_{\mathcal{BMO}(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{BMO}_w(\mathbb{R}^n)}$, which completes the proof of Proposition 2.1. \square

Lemma 2.2. *Let f be a measurable function such that $\text{supp } f \subset B := B(x_0, r)$ with $x_0 \in \mathbb{R}^n$ and $r \in (0, \infty)$. Then there exists a positive constant $C := C(\phi, n)$, depending only on ϕ and n , such that, for all $x \notin B$,*

$$\frac{1}{|x - x_0|^n} \left| \int_{B(x_0, r)} f(y) dy \right| \leq C \mathcal{M}_\phi f(x).$$

Proof. For $x \notin B(x_0, r)$ and any $y \in B(x_0, r)$, it follows that

$$\frac{|x - y|}{2|x - x_0|} < \frac{|x - x_0| + r}{2|x - x_0|} \leq 1,$$

which, together with $\phi \equiv 1$ on $B(0, 1)$, further implies that $\phi(\frac{x-y}{2|x-x_0|}) = 1$. Thus, we know that

$$\begin{aligned} \mathcal{M}_\phi f(x) &= \sup_{t \in (0, \infty)} |f * \phi_t(x)| \geq |f * \phi_{2|x-x_0|}(x)| \\ &= \frac{1}{2^n |x-x_0|^n} \left| \int_{B(x_0, r)} f(y) \phi\left(\frac{x-y}{2|x-x_0|}\right) dy \right| \\ &\gtrsim \frac{1}{|x-x_0|^n} \left| \int_{B(x_0, r)} f(y) dy \right|, \end{aligned}$$

which completes the proof of Lemma 2.2. \square

Lemma 2.3. *Let $w \in A_\infty(\mathbb{R}^n)$ and $q \in [1, \infty)$. Then there exists a positive constant C such that, for any $f \in \text{BMO}(\mathbb{R}^n)$ and any ball $B \subset \mathbb{R}^n$,*

$$\left[\frac{1}{w(B)} \int_B |f(x) - f_B|^q w(x) dx \right]^{1/q} \leq C \|f\|_{\text{BMO}(\mathbb{R}^n)}.$$

Proof. It follows from the John-Nirenberg inequality that there exist two positive constants c_1 and c_2 , depending only on n , such that, for all $\lambda > 0$,

$$|\{x \in B : |f(x) - f_B| > \lambda\}| \leq c_1 e^{-c_2 \frac{\lambda}{\|f\|_{\text{BMO}(\mathbb{R}^n)}}} |B|;$$

see [8]. Therefore, by (2.1), we see that

$$\begin{aligned} \frac{1}{w(B)} \int_B |f(x) - f_B|^q w(x) dx &= q \int_0^\infty \lambda^{q-1} \frac{w(\{x \in B : |f(x) - f_B| > \lambda\})}{w(B)} d\lambda \\ &\lesssim \int_0^\infty \lambda^{q-1} \left[\frac{|\{x \in B : |f(x) - f_B| > \lambda\}|}{|B|} \right]^{(r-1)/r} d\lambda \\ &\lesssim \int_0^\infty \lambda^{q-1} e^{-c_2 \frac{\lambda}{\|f\|_{\text{BMO}(\mathbb{R}^n)}}} d\lambda \\ &\lesssim \|f\|_{\text{BMO}(\mathbb{R}^n)}^q, \end{aligned}$$

which completes the proof of Lemma 2.3. \square

Lemma 2.4. *Let $\delta \in (0, 1]$, $q \in (1, 1 + \delta/n)$ and $w \in A_q(\mathbb{R}^n)$. Assume that T is a δ -Calderón-Zygmund operator. Then there exists a positive constant C such that, for any $b \in \text{BMO}(\mathbb{R}^n)$ and $(H_w^1(\mathbb{R}^n), q)$ -atom a related to the ball $B \subset \mathbb{R}^n$,*

$$\|(b - b_B)Ta\|_{L_w^1(\mathbb{R}^n)} \leq C \|b\|_{\text{BMO}(\mathbb{R}^n)}.$$

Proof. It suffices to show that

$$I_1 := \int_{2B} |[b(x) - b_B]Ta(x)|w(x) dx \lesssim \|b\|_{\text{BMO}(\mathbb{R}^n)}$$

and

$$I_2 := \int_{(2B)^c} |[b(x) - b_B]Ta(x)|w(x) dx \lesssim \|b\|_{\text{BMO}(\mathbb{R}^n)}.$$

Indeed, by the boundedness of T from $H_w^1(\mathbb{R}^n)$ to $L_w^1(\mathbb{R}^n)$ and from $L_w^q(\mathbb{R}^n)$ to itself with $q \in (1, 1 + \delta/n)$ (see [6, Theorem 2.8]), the Hölder inequality and Lemma 2.3, we conclude that

$$\begin{aligned} (2.2) \quad I_1 &= \int_{2B} |[b(x) - b_B]Ta(x)|w(x) dx \\ &\leq |b_{2B} - b_B| \|Ta\|_{L_w^1(\mathbb{R}^n)} + \int_{2B} |[b(x) - b_{2B}]Ta(x)|w(x) dx \\ &\lesssim \|b\|_{\text{BMO}(\mathbb{R}^n)} + \left[\int_{2B} |b(x) - b_{2B}|^{q'} w(x) dx \right]^{1/q'} \left[\int_{2B} |Ta(x)|^q w(x) dx \right]^{1/q} \\ &\lesssim \|b\|_{\text{BMO}(\mathbb{R}^n)} + [w(2B)]^{1/q'} \|b\|_{\text{BMO}(\mathbb{R}^n)} \|a\|_{L_w^q(\mathbb{R}^n)} \\ &\lesssim \|b\|_{\text{BMO}(\mathbb{R}^n)}, \end{aligned}$$

here and hereafter, $1/q' + 1/q = 1$.

On the other hand, by the Hölder inequality, (1.3), Lemma 2.3 and (2.1), we know that

$$\begin{aligned} (2.3) \quad I_2 &= \int_{(2B)^c} |[b(x) - b_B]Ta(x)|w(x) dx \\ &= \int_{(2B)^c} |b(x) - b_B| \left| \int_B a(y) [K(x, y) - K(x, x_0)] dy \right| w(x) dx \\ &\leq \int_B |a(y)| \int_{(2B)^c} |b(x) - b_B| |K(x, y) - K(x, x_0)| w(x) dx dy \\ &= \int_B |a(y)| \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} |b(x) - b_B| |K(x, y) - K(x, x_0)| w(x) dx dy \\ &\lesssim \int_B |a(y)| dy \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} \frac{r^\delta}{(2^k r)^{n+\delta}} |b(x) - b_B| w(x) dx \\ &\lesssim \left[\int_B |a(y)|^q w(y) dy \right]^{1/q} \left[\int_B [w(y)]^{-q'/q} dy \right]^{1/q'} \\ &\quad \times \sum_{k=1}^{\infty} 2^{-k\delta} \frac{1}{|2^{k+1}B|} \int_{2^{k+1}B} [|b(x) - b_{2^{k+1}B}| + |b_{2^{k+1}B} - b_B|] w(x) dx \\ &\lesssim \frac{|B|}{w(B)} \sum_{k=1}^{\infty} 2^{-k\delta} k \frac{w(2^{k+1}B)}{|2^{k+1}B|} \|b\|_{\text{BMO}(\mathbb{R}^n)} \\ &\lesssim \|b\|_{\text{BMO}(\mathbb{R}^n)} \sum_{k=1}^{\infty} k 2^{-k[\delta+n-nq]} \\ &\lesssim \|b\|_{\text{BMO}(\mathbb{R}^n)}, \end{aligned}$$

since $\delta + n - nq > 0$ and $|b_{2^{k+1}B} - b_B| \lesssim k \|b\|_{\text{BMO}(\mathbb{R}^n)}$ for all $k \geq 1$.

Combining (2.2) and (2.3), we then complete the proof of Lemma 2.4. \square

The following lemma is due to Bownik et al. [3, Theorem 7.2].

Lemma 2.5. *Let $w \in A_{1+\delta/n}(\mathbb{R}^n)$ and \mathcal{X} be a Banach space. Assume that T is a linear operator defined on the space of finite linear combinations of continuous $(H_w^1(\mathbb{R}^n), \infty)$ -atoms with the property that*

$$\sup \{ \|T(a)\|_{\mathcal{X}} : a \text{ is a continuous } (H_w^1(\mathbb{R}^n), \infty)\text{-atom} \} < \infty.$$

Then T admits a unique continuous extension to a bounded linear operator from $H_w^1(\mathbb{R}^n)$ into \mathcal{X} .

Let $w \in A_{1+\delta/n}(\mathbb{R}^n)$ and $\varepsilon \in (0, \infty)$. Recall that m is called an $(H_w^1(\mathbb{R}^n), \infty, \varepsilon)$ -molecule related to the ball $B \subset \mathbb{R}^n$ if

- (i) $\int_{\mathbb{R}^n} m(x) dx = 0$,
- (ii) $\|m\|_{L^\infty(S_j)} \leq 2^{-j\varepsilon} [w(S_j)]^{-1}$, $j \in \mathbb{Z}_+$, where $S_0 = B$ and $S_j = 2^{j+1}B \setminus 2^jB$ for $j \in \mathbb{N}$.

Lemma 2.6. *Let $w \in A_{1+\delta/n}(\mathbb{R}^n)$ and $\varepsilon > 0$. Then there exists a positive constant C such that, for any $(H_w^1(\mathbb{R}^n), \infty, \varepsilon)$ -molecule m related to the ball B , it holds true that*

$$m = \sum_{j=0}^{\infty} \lambda_j a_j,$$

where $\{a_j\}_{j=0}^{\infty}$ are $(H_w^1(\mathbb{R}^n), \infty)$ -atoms related to the balls $\{2^{j+1}B\}_{j \in \mathbb{Z}_+}$ and there exists a positive constant C such that $|\lambda_j| \leq C2^{-j\varepsilon}$ for all $j \in \mathbb{Z}_+$.

Proof. The proof of this lemma is standard (see, for example, [12, Theorem 4.7]), the details being omitted. \square

Now we are ready to give the proofs of Theorems 1.3 and 1.5.

Proof of Theorem 1.3. First, we prove that (ii) implies (i). Since $w \in A_{1+\delta/n}(\mathbb{R}^n)$, it follows that there exists $q \in (1, 1 + \delta/n)$ such that $w \in A_q(\mathbb{R}^n)$. By Lemma 2.5, it suffices to prove that, for any continuous $(H_w^1(\mathbb{R}^n), \infty)$ -atom a related to the ball $B = B(x_0, r)$ with $x_0 \in \mathbb{R}^n$ and $r \in (0, \infty)$,

$$(2.4) \quad \|[b, T](a)\|_{L_w^1(\mathbb{R}^n)} \lesssim \|b\|_{\mathcal{BMO}_w(\mathbb{R}^n)}.$$

By Lemma 2.4 and the boundedness of T from $H_w^1(\mathbb{R}^n)$ to $L_w^1(\mathbb{R}^n)$, (2.4) is reduced to showing that

$$(2.5) \quad \|(b - b_B)a\|_{H_w^1(\mathbb{R}^n)} \lesssim \|b\|_{\mathcal{BMO}_w(\mathbb{R}^n)}.$$

To do this, for every $x \in (2B)^c$ and $y \in B$, we see that $|x - y| \sim |x - x_0|$ and

$$\mathcal{M}_\phi([b - b_B]a)(x) \lesssim \sup_{t \in (0, \infty)} \frac{1}{t^n} \int_B \int_B |b(y) - b_B| |a(y)| \left| \phi\left(\frac{x - y}{t}\right) \right| dy$$

$$\lesssim \frac{1}{|x - x_0|^n} \int_B |b(y) - b_B| |a(y)| dy.$$

Hence

$$\int_{(2B)^c} \mathcal{M}_\phi([b - b_B]a)(x) w(x) dx \lesssim \|b\|_{\mathcal{BMO}_w(\mathbb{R}^n)}.$$

In addition, by the boundedness of \mathcal{M}_ϕ on $L_w^q(\mathbb{R}^n)$ with $q \in (1, 1 + \delta/n)$, Lemma 2.3 and Proposition 2.1, we know that

$$\begin{aligned} \int_{2B} \mathcal{M}_\phi([b - b_B]a)(x) w(x) dx &\lesssim w(2B)^{1/q'} \|(b - b_B)a\|_{L_w^q(\mathbb{R}^n)} \\ &\lesssim \left[\frac{1}{w(B)} \int_B |b(x) - b_B|^q w(x) dx \right]^{1/q} \\ &\lesssim \|b\|_{\text{BMO}(\mathbb{R}^n)} \\ &\lesssim \|b\|_{\mathcal{BMO}_w(\mathbb{R}^n)}, \end{aligned}$$

which concludes the proof of (ii) implying (i).

We now prove that (i) implies (ii). Let $\{R_j\}_{j=1}^n$ be the classical Riesz transforms. Then, by Lemma 2.4, we find that, for any $(H_w^1(\mathbb{R}^n), \infty)$ -atom a related to the ball B and $j \in \{1, \dots, n\}$,

$$\begin{aligned} \|R_j([b - b_B]a)\|_{L_w^1(\mathbb{R}^n)} &\leq \|[b, R_j](a)\|_{L_w^1(\mathbb{R}^n)} + \|(b - b_B)R_j a\|_{L_w^1(\mathbb{R}^n)} \\ &\lesssim \|[b, R_j]\|_{H_w^1(\mathbb{R}^n) \rightarrow L_w^1(\mathbb{R}^n)} + \|b\|_{\text{BMO}(\mathbb{R}^n)}, \end{aligned}$$

here and hereafter,

$$\|[b, R_j]\|_{H_w^1(\mathbb{R}^n) \rightarrow L_w^1(\mathbb{R}^n)} := \sup_{\|f\|_{H_w^1(\mathbb{R}^n)} \leq 1} \|[b, R_j]f\|_{L_w^1(\mathbb{R}^n)}.$$

By the Riesz transform characterization of $H_w^1(\mathbb{R}^n)$ (see [13]), we see that $(b - b_B)a \in H_w^1(\mathbb{R}^n)$ and, moreover,

$$(2.6) \quad \|(b - b_B)a\|_{H_w^1(\mathbb{R}^n)} \lesssim \|b\|_{\text{BMO}(\mathbb{R}^n)} + \sum_{j=1}^n \|[b, R_j]\|_{H_w^1(\mathbb{R}^n) \rightarrow L_w^1(\mathbb{R}^n)}.$$

For any ball $B := B(x_0, r) \subset \mathbb{R}^n$ with $x_0 \in \mathbb{R}^n$ and $r \in (0, \infty)$, let

$$a := \frac{1}{2w(B)}(f - f_B)\chi_B,$$

where $f := \text{sign}(b - b_B)$. It is easy to see that a is an $(H_w^1(\mathbb{R}^n), \infty)$ -atom related to the ball B . Moreover, for every $x \notin B$, Lemma 2.2 gives us that

$$\begin{aligned} \frac{1}{|x - x_0|^n} \frac{1}{2w(B)} \int_B |b(x) - b_B| dx &= \frac{1}{|x - x_0|^n} \int_B (b(x) - b_B)a(x) dx \\ &\lesssim \mathcal{M}_\phi([b - b_B]a)(x). \end{aligned}$$

This, together with (2.6), allows to conclude that $b \in \mathcal{BMO}_w(\mathbb{R}^n)$ and, moreover,

$$\|b\|_{\mathcal{BMO}_w(\mathbb{R}^n)} \lesssim \|b\|_{\text{BMO}(\mathbb{R}^n)} + \sum_{j=1}^n \|[b, R_j]\|_{H_w^1(\mathbb{R}^n) \rightarrow L_w^1(\mathbb{R}^n)},$$

which complete the proof of Theorem 1.3. \square

Proof of Theorem 1.5. By Lemma 2.5, it suffices to prove that, for any continuous $(H_w^1(\mathbb{R}^n), \infty)$ -atom a related to the ball B ,

$$(2.7) \quad \|[b, T](a)\|_{H_w^1(\mathbb{R}^n)} \lesssim \|b\|_{\mathcal{BMO}_w(\mathbb{R}^n)}.$$

By (2.5) and the boundedness of T on $H_w^1(\mathbb{R}^n)$ (see [9, Theorem 1.2]), (2.7) is reduced to proving that

$$\|(b - b_B)Ta\|_{H_w^1(\mathbb{R}^n)} \lesssim \|b\|_{\mathcal{BMO}_w(\mathbb{R}^n)}.$$

Since $w \in A_{1+\delta/n}(\mathbb{R}^n)$, it follows that there exists $q \in (1, 1+\delta/n)$ such that $w \in A_q(\mathbb{R}^n)$. By this and the fact that T is a δ -Calderón-Zygmund operator, together with a standard argument, we find that Ta is an $(H_w^1(\mathbb{R}^n), \infty, \varepsilon)$ -molecule related to the ball B with $\varepsilon := n + \delta - nq > 0$. Therefore, by Lemma 2.6, we have

$$Ta = \sum_{j=0}^{\infty} \lambda_j a_j,$$

where $\{a_j\}_{j=0}^{\infty}$ are $(H_w^1(\mathbb{R}^n), \infty)$ -atoms related to the balls $\{2^{j+1}B\}_{j=0}^{\infty}$ and $|\lambda_j| \lesssim 2^{-j\varepsilon}$ for all $j \in \mathbb{Z}_+$. Thus, by (2.5) and Proposition 2.1, we obtain

$$\begin{aligned} \|(b - b_B)Ta\|_{H_w^1(\mathbb{R}^n)} &\leq \sum_{j=0}^{\infty} |\lambda_j| [\|(b - b_{2^{j+1}B})a_j\|_{H_w^1(\mathbb{R}^n)} + \|(b_{2^{j+1}B} - b_B)a_j\|_{H_w^1(\mathbb{R}^n)}] \\ &\lesssim \|b\|_{\mathcal{BMO}_w(\mathbb{R}^n)} \sum_{j=0}^{\infty} 2^{-j\varepsilon} + \|b\|_{\text{BMO}(\mathbb{R}^n)} \sum_{j=0}^{\infty} (j+1)2^{-j\varepsilon} \\ &\lesssim \|b\|_{\mathcal{BMO}_w(\mathbb{R}^n)}, \end{aligned}$$

which completes the proof of (i) implying (ii) and hence Theorem 1.5. \square

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